DIFFERENTIAL GEOMETRY, PALATINI GRAVITY AND REDUCTION

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Abstract. The present article deals with a formulation of the so called *(vacuum) Palaltini gravity* as a general variational principle. In order to accomplish this goal, some geometrical tools related to the geometry of the bundle of connections of the frame bundle LM are used. A generalization of Lagrange-Poincaré reduction scheme to these types of variational problems allows us to relate it with the Einstein-Hilbert variational problem. Relations with some other variational problems for gravity found in the literature are discussed.

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1. Notations

The internal metric of the tetrads will have the signature $(- + \cdots +)$. We will assume further the conventions of [MR94] in playing with forms. If α is a $k-$form:

$$X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \beta)$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

$$(d\alpha)(X_0, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i X_i \cdot \left(\alpha\left(X_0, \cdots, \widehat{X_i}, \cdots, X_k\right)\right) + $$
$$+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha\left([X_i, X_j], X_0, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_k\right)$$

The indices $\mu, \nu, \cdots$ and $i, j, k, \cdots$ will run from 1 to n; as usual, the first set will be used in the enumeration of local coordinates on spacetime, while the latin indices will label the components in

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the (tensorial algebra of the) local model $\mathbb{R}^n$. In particular, we are using the following convention relating group product in $GL(n)$ and indices

$$(g \cdot h)_i^j = g_i^k h_k^j$$

for all $g, h \in GL(n)$. Following standard usage, we will use the acronym EDS when refering to *exterior differential systems*.

2. Introduction

The purpose of this work is to characterize geometrically the variational problems associated with the collection of physical theories known as *(vacuum) Palatini gravity*. In these theories the usual field for gravity, namely the metric, is replaced by a connection and collection of local basis for the tangent bundle to the spacetime, known as *vielbeins*. First order variational principles along these lines were first described in the groundbreaking work of Arnowitt, Deser and Misner [ADM04] in terms of connection variables and metric, and within the vierbein formalism in [DI76]. In accomplishing this task, it is necessary not only to identify the correct mathematical structure describing the fields, but to agree on the meaning of "variational problem". The usual way to describe a field theory uses the following ingredients: A bundle $E \to M$ and a Lagrangian density $L : J^k E \to \Omega^n(M)$, where $n = \dim M$. As mentioned above, when describing gravity as a field theory, we can use the metric as a fundamental degree of freedom, in such a way that $E = S(T^*M \otimes T^*M)$ and $k = 2$ (although this is the setting for the *Einstein-Hilbert gravity*, a first order formulation is provided in [ADM04], and an alternate variational principle equivalent to this one will be presented in the present work), or these degrees of freedom can correspond to a metric, a frame and a connection, as in the case of the so called *metric-affine gravity* [HMMN95] or the $GL(4)$*-invariant gravity* [FP90b]. Following [Got91], the variational problems we will consider here are composed of three data: A bundle on the spacetime, a Lagrangian form on it and a restriction posed in terms of an exterior differential system. In the usual approach to field theory, the underlying bundle is $J^k E \to M$ ($E \to M$ is the bundle whose sections are the fields) and the restriction EDS is induced by the contact structure of the jet bundle; it relates the different degrees of freedom of the theory, in such a way that some of them are derivatives of the remaining. Thus the concept of variational problem adopted in this work includes not only the usual field theory, but instances in which the relation between fields are of different nature: In fact, our description of Palatini gravity will use this additional freedom in order to properly encode the requeriments of metricity and torsionless. Thus, in the following, the words *variational problem* (in this sense) and *field theory* will be used interchangeably.

Now, in order to find a formulation for the field theory capturing the essential ingredients of Palatini gravity, we will try to use as fields the metric on spacetime and a connection on it, as in the metric-affine approach; nevertheless, it must be stressed that we will use local vielbeins in order to specify the metric. So, in this setting the degrees of freedom are local frames and local connection forms, and the metric is determined from them; the Lagrangian of the theory is obtained by writing out the scalar curvature in terms of these geometric data; for a review see [Pel94]. Namely, if $\left(e_k, \omega_j^i\right)$ is a pair composed of a local basis (e_k) for the tangent bundle of the spacetime M and the set $\left(\omega_i^j\right)$ of 1-forms provides the local description for the covariant operator according to the formula

$$\nabla e_i := \omega_i^j e_j,$$

the Palatini variational problem consists into the variations of the action

$$S[e, \omega] := \int_M \epsilon_{ijkl} \eta^{ip} \theta^k \wedge \theta^l \wedge \left(\mathbf{d}\omega_p^j + \omega_q^j \wedge \omega_p^q\right)$$

taking as granted that the basis (e_k) and the forms $\left(\omega_j^i\right)$ can be varied independently; $\left(\theta^k\right)$ is the dual basis to (e_j). Because of the compatibility relation for two local descriptions $\left(e_k, \omega_j^i\right)$ and $\left(\bar{e}_k, \bar{\omega}_j^i\right)$ of the same connection, namely

$$\bar{\omega}_j^i = h_k^i \mathbf{d} g_j^k + h_k^i \omega_l^k g_j^l,$$

if and only if

$$\bar{e}_k = g_k^l e_l, \qquad h = g^{-1},$$

this version of the variational problem appears to be dissapointing in two main aspects:

(1) the variations could have some relation between them, and
(2) the action S extends to a global action (i.e, an action depending only on the connection and not in the local description) if and only if $g \in SO(1, n-1)$; that is, iff the local sections $m \mapsto (e_k(m))$ are sections of a subbundle $L^h M$ of the frame bundle LM with structure group $SO(1, n-1)$. This subbundle can be seen as the set of orthonormal basis with respect to a metric h, that can be recovered via $h = \eta^{kl} e_k \otimes e_l$.

These concerns are triggered by an inadequate choice of the fundamental degrees of freedom of the theory: As we will show below (see Section 4.1.5), on the jet bundle $J^1 LM$ of the frame bundle there exists a true $\mathfrak{gl}(n)$-valued 1-form whose pullback along connections (in a proper sense) gives rise to the local connection forms. The local version of the Palatini Lagrangian will be nothing but the pullback of this global form along sections of $J^1 LM$.

The first part of the article will be devoted to set a bundle such that their local sections can be identified with the chosen degrees of freedom for Palatini gravity, namely, local frames and local connection forms; accordingly, we need to provide a Lagrangian form on this bundle yielding to the same action of the Palatini gravity and a set of differential restrictions playing the role of the contact structure in this case. We will use some geometrical constructions for accomplish these goals, related to the geometry of the frame bundle LM and its jet bundle. In the second part of the present work, we will further relate this description of the Palatini gravity with the variational problem associated to Einstein-Hilbert gravity. In order to carry out this task, it will be necessary to generalize the usual notion of reduction for field theory, in order to eliminate the extra degrees of freedom associated to the arbitrary local frame.

3. Variational problems and field theory

Our initial data will be a *variational triple*, that is, a triple $(\Lambda \to M, \lambda, \mathcal{I})$ composed of a fibre bundle Λ on the spacetime M, a n-form λ on it (where $n = \dim M$) and an EDS $\mathcal{I} \subset \Omega^\bullet(\Lambda)$. The bundle consists of the degree of freedom associated to the fields and its (generalized) velocities, the n-form λ will be used to define the dynamics, and $\mathcal{I}$ will induce some relations between the degrees of freedom (for example, it will force to some variables to be the derivatives of another variables).

3.1. Variational problems.

We are in position to formulate a notion of *variational problems*. Although we are primarily interested in applications of these notions to physics, they can be used in tackling geometrical problems, see [Hsu92].

Definition 1. *The* variational problem *associated to a variational triple* $(\Lambda, \lambda, \mathcal{I})$ *consists into the problem of finding the sections* $\sigma : M \to \Lambda$ *which are integrals for the EDS* $\mathcal{I}$ *and extremals for the functional*

$$S[\sigma] := \int_M \sigma^* \lambda.$$

Note 2. We will suppose that the necessary conditions for the existence of the several integrals that could appear throughout the work are met; for example, M would be compact.

Definition 3 (Infinitesimal symmetries of an EDS)**.** *Let* $\mathcal{I} \subset \Omega^\bullet(\Lambda)$ *be an EDS. A (perhaps local) vector field* X *is an* infinitesimal symmetry of $\mathcal{I}$ *if and only if*

$$\mathcal{L}_X \mathcal{I} \subset \mathcal{I}.$$

The set of infinitesimal symmetries of $\mathcal{I}$ *will be indicated by* $\mathsf{Symm}(\mathcal{I})$.

Definition 4 (Euler-Lagrange EDS)**.** *Let* $(\Lambda, \lambda, \mathcal{I})$ *be a variational problem. The* Euler-Lagrange EDS *is the EDS generated by the set of forms*

$$\left\{\alpha \in \Omega^\bullet(\Lambda) : \alpha - X \lrcorner \boldsymbol{d}\lambda \equiv 0 \mod \boldsymbol{d}\Omega^{n-1}(\Lambda) \text{ for all } X \in \mathsf{Symm}(\mathcal{I}) \cap \mathfrak{X}^V(\Lambda)\right\}.$$

3.2. **Classical field theory as variational problem.** It is necessary perhaps to indicates the way in which the usual (first order) classical field theory fits in this scheme: The corresponding variational problem is simply $\left(J^1E, \mathcal{L}\mathbf{d}x^1\wedge\cdots\wedge\mathbf{d}x^n, \mathcal{I}_{\text{con}}\right)$, where $E\to M$ is a bundle on M (whose nature is associated to the field to be described by the theory), $\mathcal{L}$ is the Lagrangian density of the theory and

$$\mathcal{I}_{\text{con}} := \left\langle \mathbf{d}u^A - u^A_k\mathbf{d}x^k\right\rangle_{\text{diff}}$$

is the contact structure of the jet space. This variational problem is usually called in the literature the *classical variational problem* [Got91, Gri98]. Then we have the following result.

Lemma 5. *The underlying PDE for the Euler-Lagrange EDS associated to the classical variational problem contains the Euler-Lagrange equations.*

Proof. Let us work in local coordinates. For $X := \left(0, \delta u^A, \delta u^A_k\right) \in \mathsf{Symm}\left(\mathcal{I}_{\text{con}}\right)\cap\mathfrak{X}^V\left(\Lambda\right)$, we have that

$$\mathbf{d}\delta u^A - \delta u^a_k\mathbf{d}x^k = 0;$$

then

$$X\lrcorner\mathbf{d}\lambda = \left(\frac{\partial\mathcal{L}}{\partial u^A}\delta u^A + \frac{\partial\mathcal{L}}{\partial u^A_k}\delta u^A_k\right)\mathbf{d}x^1\wedge\cdots\wedge\mathbf{d}x^n.$$

Let $\sigma_A\in\Omega^{n-1}\left(J^1E\right)$ be defined as

$$\sigma_A := \left(\frac{\partial\mathcal{L}}{\partial u^A_k}\frac{\partial}{\partial x^k}\right)\lrcorner\mathbf{d}x^1\wedge\cdots\wedge\mathbf{d}x^n;$$

therefore

$$\begin{aligned}
X\lrcorner\mathbf{d}\lambda &= \frac{\partial\mathcal{L}}{\partial u^A}\delta u^A\mathbf{d}x^1\wedge\cdots\wedge\mathbf{d}x^n + \sigma_A\wedge\delta u^A_k\mathbf{d}x^k\\
&= \frac{\partial\mathcal{L}}{\partial u^A}\delta u^A\mathbf{d}x^1\wedge\cdots\wedge\mathbf{d}x^n + \sigma_A\wedge\mathbf{d}\delta u^A\\
&\equiv \left[(-1)^{n+1}\mathbf{d}\sigma_A + \frac{\partial\mathcal{L}}{\partial u^A}\mathbf{d}x^1\wedge\cdots\wedge\mathbf{d}x^n\right]\delta u^A \mod \mathbf{d}\Omega^{n-1}\left(J^1E\right),
\end{aligned}$$

and so

$$\alpha_A := (-1)^{n+1}\mathbf{d}\sigma_A + \frac{\partial\mathcal{L}}{\partial u^A}\mathbf{d}x^1\wedge\cdots\wedge\mathbf{d}x^n$$

are generators for the Euler-Lagrange EDS. Any integral section for this EDS will obey the Euler-Lagrange equations associated to $\mathcal{L}$. □

4. Some tools from differential geometry

It is time to introduce the basic language we will use to describe gravitation in this work; it will be necessary to point out here some useful tools borrowed from differential geometry in the handling of the multiple questions raised while working with connections.

In the present section we will describe briefly differential geometry from moving frame viewpoint, as in [KN63, Spi79]. Whenever possible, we will make contact with the more usual description in terms of principal bundles; this framework is of outmost importance in the description of Palatini gravity in the present work. So we will need some facts concerning the jet bundle of a principal bundle. This is a natural choice in this context, because the first structure equation on LM allow us to consider a connection as a kind of velocity associated to the degree of freedom provided by a frame. From this point of view, we need a set of forms on J^1LM encoding the structure equations; it results from the work of García [Gar72] and Castrillón *et al.* [CLMnM01] that there exists a $\mathfrak{gl}\left(n\right)$-valued 2-form on J^1LM such that its pullback along a connection (in an appropiate sense, see below for details) is the curvature of this connection. Additionally, it can be defined a $\mathbb{R}^n$-valued 2-form giving rise to the torsion of the connection via the same pullback procedure. These forms are the fundamental ingredients in the construction of the equivalent of Palatini Lagrangian in this context.

4.1. **The geometry of J^1P.** The following section has been formulated by making heavy use of the reference [CLMnM01]; it can be considered as a natural continuation of the last example in [Mar05] to this context.

4.1.1. *Geometric preliminaries.* Let $\tau : P \to M$ be a G-principal bundle on M; then we have the diagram

$$\begin{tikzcd} TP/G \arrow[r, "T\tau"] \arrow[d, "\tau_M^P"'] & TM \arrow[d, "\tau_M"] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{1}$$

where it was defined

$$\tau_M^P\left([v]_G\right) := \tau\left(\tau_P\left(v\right)\right).$$

Definition 6. *The bundle of connections $C(P)$ is the bundle on M given by*

$$C(P) := \left\{\lambda : T_mM \to \left.(TP/G)\right|_m \text{ such that } T\tau \circ \lambda = id_{T_mM.}\right\}.$$

The main tool to work with this bundle is the following lemma; it relies essentially in the fact that the G-orbits are vertical, and the action is free.

Lemma 7. *There exists a bundle isomorphism between $C(P)$ and J^1P/G.*

The bundle isomorphism between $C(P)$ and J^1P/G is proved in [CLMnM01] using the fact that there exists a G-principal bundle structure $q : J^1P \to C(P)$ through the right G-action determined by the lift of the G-action on P. If we consider the 1-jet space as the set

$$J^1P = \bigcup_{p\in P}\left\{\rho : T_{\tau(p)}M \to T_pP \text{ such that } T_p\tau \circ \rho = \mathrm{id}_{T_{\tau(p)}M}\right\},$$

then $q(\rho) := p_G \circ \rho$, where $p_G : TP \to TP/G$ is the canonical projection for the quotient; it is convenient at this point to remember that the 1-jet bundle J^1P comes with the maps fitting in the diagram

$$\begin{tikzcd} J^1P \arrow[rr, "\tau_{10}"] \arrow[dr, "\tau_1"'] & & P \arrow[dl, "\tau"] \\ & M & \end{tikzcd}$$

This identification allows us to use the map $\tau_1 : J^1P \to M$ as the fibre bundle map of $C(P)$ on M. On the other side, every element $[\rho]_G \in C(P)$ can be thought as a "connection form at $m := \tau_1\left([\rho]_G\right)$", as the following proposition shows.

Proposition 8. *Every element $[\rho]_G$ defines a unique family of projections $\Gamma_p : T_pP \to V_pP$ for $p \in \tau^{-1}(m)$.*

Proof. In fact, for $p \in \tau^{-1}(m)$, we can define the projection map

$$\Gamma_p := T_{\tau_{10}(\rho)}R_g \circ \Gamma_{\tau_{10}(\rho)} \circ T_pR_{g^{-1}},$$

if and only if $p = \tau_{10}(\rho)\, g$ and

$$\Gamma_{\tau_{10}(\rho)} := \mathrm{id}_{T_{\tau_{10}(\rho)}P} - \rho \circ T_{\tau_{10}(\rho)}\tau.$$

Namely, we select an element $\rho \in [\rho]_G$ and define on $p_0 := \tau_{10}(\rho) \in P$ a projection $\Gamma_{p_0} : T_{p_0}P \to V_{p_0}P$ onto the vertical fibre; then we extend this projection to any point of $\tau^{-1}(m)$ by using the right G-action. Because of the form we choose to do this extension, it results that this definition is independent of the choice made of the representative $\rho \in [\rho]_G$. $\square$

4.1.2. *The universal form on J^1P.* We can use Proposition 8 in order to construct a $\mathfrak{g}$-valued 1-form on J^1P; it is necessary first to recall that we have a vector bundle isomorphism

$$P\times\mathfrak{g}\to VP:(p,\xi)\mapsto\left.\frac{\vec{\mathrm{d}}}{\mathrm{d}t}\right|_{t=0}[p\cdot(\exp t\xi)].$$

Then the canonical connection $\omega\in\Omega^1\left(J^1P,\mathfrak{g}\right)$ is defined through

$$\left.\omega\right|_\rho(Y):=[\rho]_G\left(T_\rho\tau_{10}(Y)\right)$$

where $[\rho]_G\in C(P)$ has been considered in this formula as the family of projections $\Gamma_{\tau_{10}(\rho)}$ of the previous proposition, and we freely use the identification $V_{\tau_{10}(\rho)}P\simeq\mathfrak{g}$.

Lemma 9. *The 1-form ω generates the contact structure on P.*

Proof. Let $s:M\to P$ be a local section of P; then we have that

$$\tau_{10}\circ j^1s=s$$

where

$$j^1s:M\to J^1P:m\mapsto[v\in T_mM\mapsto(T_ms)(v)]$$

is the 1-jet of s. So if $v\in T_mM$ and $Y=T_m\left(j^1s\right)(v)$, we will have that

$$T_{j^1_ms}\tau_{10}(Y)=T_ms(v);$$

therefore

$$\begin{aligned}\left.\omega\right|_{j^1_ms}(Y)&=\left[j^1_ms\right]_G\left(T_ms(v)\right)\\&=T_ms(v)-j^1_ms\circ T_{s(m)}\tau\left(T_ms(v)\right)\\&=T_ms(v)-j^1_ms(v)\\&=0.\end{aligned}$$

It means that ω is in the algebraic closure of the contact forms. □

Thus we have the following result [CLMnM01].

Proposition 10. *The bundle $p_G:J^1P\to J^1P/G=C(P)$ is a G-principal bundle, and ω defines a connection on it.*

We can form now the pullback bundle

$$\begin{tikzcd}\tau_1^*\left(\mathrm{ad}(P)\right) \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathrm{ad}(P) \arrow[d] \\ C(P) \arrow[r, "\tau_1"'] & M\end{tikzcd}$$

where $\mathrm{ad}(P):=(P\times\mathfrak{g})/G$, taking $\mathfrak{g}$ as a G-space through the adjoint action. Then there exists a $\mathfrak{g}$-valued 2-form Ω of the adjoint type on J^1P, namely the curvature form associated to the connection ω; it defines a 2-form on $C(P)$ with values in $\tau_1^*\left(\mathrm{ad}(P)\right)$ via

$$\left.\Omega_2\right|_{[\rho]_G}\left(T_\rho\tau_1(X),T_\rho\tau_1(Y)\right):=\left[\rho,\left.\Omega\right|_\rho(X,Y)\right]_G$$

for $X,Y\in T_\rho J^1P$.

4.1.3. *The form ω as a universal connection.* We will prove here that the form ω can be considered as a "universal form", namely, that every connection on P can be built as a pullback of it along a suitable map.

Proposition 11. *The G-principal bundle $p_G : J^1P \to J^1P/G$ is isomorphic to p^*P.*

Proof. The bundle p^*P is defined through the diagram

$$\begin{tikzcd} p^*P \arrow[r, "p_2"] \arrow[d, "p_1"'] & P \arrow[d, "\tau"] \\ J^1P/G \arrow[r, "p"'] & M \end{tikzcd}$$

where p_1, p_2 are the projections onto the factors of the cartesian product ($p^*P \subset J^1P/G \times P$), and

$$p\left(\left[j_x^1 s\right]_G\right) := x.$$

The bundle isomorphism is defined by

$$j_x^1 s \in J^1P \mapsto \left(\left[j_x^1 s\right]_G, s(x)\right) \in J^1P/G \times P,$$

whose range is in p^*P, because

$$p\left(\left[j_x^1 s\right]_G\right) = x = \tau\left(s(x)\right).$$

In order to show that it is a diffeomorphism, it is enough to show an inverse map, namely

$$\left(\left[j_x^1 s\right]_G, u\right) \in p^*P \mapsto j_x^1 \tilde{s}$$

where $\tilde{s} : U_x \subset M \to P$ is a local section for P defined in a neighborhood U_x of x such that $\tilde{s}(x) = u$ and

$$\left[j_x^1 \tilde{s}\right]_G = \left[j_x^1 s\right]_G.$$

It is clear that such a section exists, by defining $\tilde{s}(y) = s(y) \cdot g_0$ for $y \in U_x$ and $g_0 \in G$ such that $u = s(x) \cdot g_0$. Moreover, it is a well-defined map, and this can be seen as follows: If $\tilde{t}$ is another local section verifying that $\tilde{t}(x) = u$, then there exists $\gamma : U_x \to G$ such that

$$\tilde{t}(y) = s(y) \cdot \gamma(y)$$

for all $y \in U_x$, and in particular $\gamma(x) = g_0$; so

$$j_x^1 \tilde{t} : v \in T_xM \to T_x\tilde{s}(v) = T_xs(v) \cdot g_0 + \left[\left(L_{g_0*}\right)^{-1} T_x\gamma(v)\right]_u^P \in T_uP, \tag{2}$$

where $\xi_u^P \in V_uP$ indicates the infinitesimal generator for the G-action on P associated to the element $\xi \in \mathfrak{g}$. Therefore from the condition $\left[j_x^1 \tilde{t}\right]_G = \left[j_x^1 s\right]_G$ we obtain that

$$T_x\tilde{t}(v) = T_xs(v) \cdot g_1$$

for some $g_1 \in G$, and thus must be $g_1 = g_0$, because $T_xs(v) \cdot g_1 \in T_{s(x)\cdot g_1}$ and $T_x\tilde{t}(v) \in T_uP = T_{s(x)\cdot g_0}P$. Therefore

$$\left[\left(L_{g_0*}\right)^{-1} T_x\gamma(v)\right]_u^P = 0$$

and using Eq. (2), $j_x^1 \tilde{t} = j_x^1 s \cdot g_0 = j_x^1 \tilde{s}$. ☐

According to Prop. 8, every section of $p : C(P) \to M$ defines a connection on P and conversely, every connection gives rise to a section of the bundle $C(P)$; by using Prop. 11, we can state the following result.

Proposition 12. *Every connection Γ determines a section of the affine bundle $\tau_{10} : J^1P \to P$.*

Proof. We will denote by $\sigma_\Gamma : M \to C(P)$ the section associated by Prop. 8 to the connection Γ; thus we define the map

$$\tilde{\sigma}_\Gamma(u) : P \to C(P) \times P : u \mapsto (\sigma_\Gamma(\tau(u)), u).$$

But it is immediate to show that its range is in p^*P, because of the identity

$$p(\sigma_\Gamma(u)) = \tau(u);$$

it is additionally a section of $\tau_{10} : J^1P \to P$, because under the bundle identification $J^1P \simeq p^*P$ the map τ_{10} reduces to

$$\tau_{10}\left(\left[j^1_x s\right]_G, u\right) = u,$$

and the proposition follows. ☐

We are ready to formulate the universal property of ω.

Proposition 13. *For every connection Γ on P we have that $\tilde{\sigma}^*_\Gamma\omega = \omega_\Gamma$, where $\omega_\Gamma \in \Omega^1(P, \mathfrak{g})$ is the connection form associated to Γ.*

Proof. For $X \in T_uP, x = \tau(u)$ we have that

$$\begin{aligned}\left.(\tilde{\sigma}^*_\Gamma\omega)\right|_u(X) &= \left.\omega\right|_{\tilde{\sigma}_\Gamma(u)}(\tilde{\sigma}_{\Gamma*}(X)) \\ &= \left.\omega\right|_{\tilde{\sigma}_\Gamma(u)}((T_x\sigma_\Gamma)(T_u\tau(X)), X).\end{aligned}$$

Now we have that the definition of ω involves the projection $T_{j^1_x s}\tau_{10} : T_{j^1_x s}J^1P \to T_uP$, namely

$$\left.\omega\right|_{j^1_x s}(Z) = \left[j^1_x s\right]_G\left(T_{j^1_x s}\tau_{10}(Z)\right)$$

for every $Z \in T_{j^1_x s}J^1P$; under the identification $J^1P \simeq p^*P$ we have that τ_{10} is the projection onto the second factor, so

$$\left.\omega\right|_{\tilde{\sigma}_\Gamma(u)}((T_x\sigma_\Gamma)(T_u\tau(X)), X) = [\tilde{\sigma}_\Gamma(u)]_G(X) = \sigma_\Gamma(u)(X) = \left.\omega_\Gamma\right|_u(X).$$

Then $\tilde{\sigma}^*_\Gamma\omega = \omega_\Gamma$, as we want to show. ☐

4.1.4. *Canonical forms on J^1LM.* The previous paragraphs tell us that ω is a canonical $\mathfrak{gl}(n)$-valued pseudotensorial 1-form of type $(GL(n), \mathrm{ad})$ defined on J^1LM. As we know, on LM there exists another canonical form, namely, the *tautological form* $\bar{\theta}$ [KN63]; thus the projection $\tau_{10} : J^1LM \to LM$ can be used in order to define a new canonical form on J^1LM, that is

$$\theta := \tau^*_{10}\bar{\theta} \in \Omega^1\left(J^1LM, \mathbb{R}^n\right).$$

Let us recall that under the identification $J^1LM \simeq p^*LM$ the canonical right action translates into

$$(\rho, u) \cdot h = (\rho, u \cdot h).$$

Using this fact, we can see that the form θ has the following remarkable properties.

Proposition 14. *The form θ is a tensorial 1-form of type $(GL(n), \mathbb{R}^n)$. Moreover, for every connection Γ on LM, we have that*

$$\tilde{\sigma}^*_\Gamma\theta = \bar{\theta}.$$

Proof. The second assumption follows easily from the definition of $\tilde{\sigma}_\Gamma$. Now let h be an element of $GL(n)$; every element $Z \in T_{(\rho,u)}J^1LM$ can be written as $Z = (V, X)$, where $\tau_1(\rho) = \tau(u)$ and moreover

$$T_\rho\tau_1(V) = T_u\tau(X).$$

Thus we have that

$$\begin{aligned}\left(R_h^*\theta\right)\big|_{(\rho,u)}(Z) &= \theta|_{(\rho,u\cdot h)}\left(T_{(\rho,u)}R_h(V,X)\right)\\ &= \theta|_{(\rho,u\cdot h)}\left((V,T_uR_hX)\right)\\ &= \bar{\theta}\big|_{u\cdot h}(T_uR_hX)\\ &= h^{-1}\cdot\left(\bar{\theta}\big|_u(X)\right)\\ &= h^{-1}\cdot\left(\theta|_{(\rho,u)}(V,X)\right)\end{aligned}$$

and θ is pseudotensorial of type $(GL(n),\mathbb{R}^n)$. Finally, to show that θ is tensorial, we need to prove that $\theta(Z)=0$ if Z is a vertical vector in $J^1LM\to J^1LM/GL(n)$. But

$$V_{(\rho,u)}\left(J^1LM\right)=\left\{(V,0)\in T_\rho\left(J^1LM/GL(n)\right)\times T_uP:T_\rho\tau_1(V)=0\right\},$$

and so, in particular, $\theta(Z)=0$ for every $Z\in VJ^1LM$. □

Then if $T\in\Omega^2\left(J^1LM,\mathbb{R}^n\right)$ is the exterior covariant differential of θ obtained using the connection ω, we will have that

$$T=\mathbf{d}\theta+\omega\overset{\wedge}{\cdot}\theta.$$

Let Γ be a connection on LM; by pulling back this expression along $\tilde{\sigma}_\Gamma$, we have that

$$\tilde{\sigma}_\Gamma^*T=\mathbf{d}\bar{\theta}+\omega_\Gamma\overset{\wedge}{\cdot}\bar{\theta}$$

which is in turn equal to the torsion T_Γ of the connection Γ. Thus T can be called *universal torsion.*

4.1.5. *Local expressions.* It is time to describe locally these constructions, in order to find expressions in the coordinates usually found in the literature; we will use the references [KN63, Nak96, Hus75] in this task.

Let $U\subset M$ be a coordinate neighborhood and $p:LM\to M$ the canonical projection of the frame bundle; on $p^{-1}(U)$ can be defined the coordinate functions

$$u\in p^{-1}(U)\mapsto\left(x^\mu(u),e_k^\nu(u)\right)$$

where $x^\mu\equiv x^\mu\circ p$ and

$$u=\left\{e_1^\mu\left.\frac{\partial}{\partial x^\mu}\right|_{p(u)},\cdots,e_n^\mu\left.\frac{\partial}{\partial x^\mu}\right|_{p(u)}\right\}.$$

If $\bar{U}\subset M$ is another coordinate neighborhood such that $U\cap\bar{U}\neq\emptyset$ and $u\in U\cap\bar{U}$, then

$$u=\left\{\bar{e}_1^\mu\left.\frac{\partial}{\partial \bar{x}^\mu}\right|_{p(u)},\cdots,\bar{e}_n^\mu\left.\frac{\partial}{\partial \bar{x}^\mu}\right|_{p(u)}\right\},$$

and the coordinates change on $p^{-1}(U)\cap p^{-1}\left(\bar{U}\right)\subset LM$ can be given as

$$\begin{aligned}\bar{x}^\mu&=\bar{x}^\mu\left(x^1,\cdots,x^n\right)\\ \bar{e}_k^\mu&=\frac{\partial\bar{x}^\mu}{\partial x^\nu}e_k^\nu.\end{aligned}$$

If we change the adapted coordinates according to the rule $\left(x^\mu,u^A\right)\mapsto\left(\bar{x}^\nu(x),\bar{u}^B(x,u)\right)$ on E, the induced coordinates on J^1E transform accordingly [Sau89]

$$\bar{u}_\mu^A=\left(\frac{\partial\bar{u}^A}{\partial u^B}u_\nu^B+\frac{\partial\bar{u}^A}{\partial x^\nu}\right)\frac{\partial x^\nu}{\partial\bar{x}^\mu}.$$

By supposing that the induced coordinates on J^1LM are in the present case $(x^\mu,e_k^\mu,e_{k\nu}^\mu)$ and $(\bar{x}^\mu,\bar{e}_k^\mu,\bar{e}_{k\nu}^\mu)$, we will have that

$$\bar{e}_{k\nu}^\mu=\left(\frac{\partial\bar{x}^\mu}{\partial x^\sigma}e_{k\rho}^\sigma+\frac{\partial^2\bar{x}^\mu}{\partial x^\rho\partial x^\sigma}e_k^\sigma\right)\frac{\partial x^\rho}{\partial\bar{x}^\nu}.$$

Take note on the fact that the functions

$$\Gamma^\sigma_{\mu\nu} := -e^\sigma_{k\nu} e^k_\mu,$$

where the quantities e^k_μ are uniquely determined by the conditions

$$e^k_\mu e^\nu_k = \delta^\nu_\mu,$$

transform accordingly to

$$\bar{\Gamma}^\mu_{\rho\gamma} = -\frac{\partial \bar{x}^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial \bar{x}^\alpha} e^\alpha_{k\sigma}\bar{e}^k_\rho - \frac{\partial^2 \bar{x}^\mu}{\partial x^\rho \partial x^\alpha}\frac{\partial x^\alpha}{\partial \bar{x}^\gamma}.$$

But by using the previous definition, we can find the way in which e^k_μ and $\bar{e}^k_\mu$ are related, namely

$$\bar{e}^k_\mu = \frac{\partial x^\gamma}{\partial \bar{x}^\mu} e^k_\gamma$$

and therefore

$$\bar{\Gamma}^\mu_{\delta\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\sigma}\frac{\partial x^\rho}{\partial \bar{x}^\nu}\frac{\partial x^\gamma}{\partial \bar{x}^\delta}\Gamma^\sigma_{\gamma\rho} - \frac{\partial^2 \bar{x}^\mu}{\partial x^\rho \partial x^\gamma}\frac{\partial x^\rho}{\partial \bar{x}^\nu}\frac{\partial x^\gamma}{\partial \bar{x}^\delta},$$

which is the transformation rule for the Christoffel symbols, if the following identity

$$\frac{\partial^2 \bar{x}^\sigma}{\partial x^\rho \partial x^\gamma}\frac{\partial x^\rho}{\partial \bar{x}^\mu}\frac{\partial x^\gamma}{\partial \bar{x}^\nu} = -\frac{\partial^2 x^\rho}{\partial \bar{x}^\mu \partial \bar{x}^\nu}\frac{\partial \bar{x}^\sigma}{\partial x^\rho}.$$

is used. So we are ready to calculate local expressions for the previously introduced canonical forms. First we have that

$$\theta^k = e^k_\mu \mathbf{d}x^\mu$$

determines the components of the tautological form on J^1LM, and the canonical connection form will result

$$\omega^k_l = e^k_\mu \left(\mathbf{d}e^\mu_l - e^\mu_{l\sigma}\mathbf{d}x^\sigma\right).$$

It is immediate to show that

$$\bar{\theta}^k = \theta^k,$$

and moreover

$$\begin{aligned}
\bar{\omega}^k_l &= \bar{e}^k_\mu \left(\mathbf{d}\bar{e}^\mu_l - \bar{e}^\mu_{l\nu}\mathbf{d}\bar{x}^\nu\right)\\
&= \frac{\partial x^\gamma}{\partial \bar{x}^\mu} e^k_\gamma \left[\mathbf{d}\left(\frac{\partial \bar{x}^\mu}{\partial x^\gamma} e^\gamma_l\right) - \left(\frac{\partial \bar{x}^\mu}{\partial x^\sigma} e^\sigma_{l\rho} + \frac{\partial^2 \bar{x}^\mu}{\partial x^\rho \partial x^\sigma} e^\sigma_l\right)\frac{\partial x^\rho}{\partial \bar{x}^\nu}\mathbf{d}\bar{x}^\nu\right]\\
&= \frac{\partial x^\gamma}{\partial \bar{x}^\mu} e^k_\gamma \left(\frac{\partial \bar{x}^\mu}{\partial x^\gamma}\mathbf{d}e^\gamma_l - \frac{\partial \bar{x}^\mu}{\partial x^\sigma} e^\sigma_{l\rho}\mathbf{d}x^\rho\right)\\
&= e^k_\gamma \left(\mathbf{d}e^\gamma_l - e^\gamma_{l\rho}\mathbf{d}x^\rho\right)\\
&= \omega^k_l.
\end{aligned}$$

The associated curvature form can be calculated according to the formula

$$\begin{aligned}
\Omega^k_l &:= \mathbf{d}\omega^k_l + \omega^k_p \wedge \omega^p_l\\
&= \mathbf{d}\left[e^k_\gamma \left(\mathbf{d}e^\gamma_l - e^\gamma_{l\rho}\mathbf{d}x^\rho\right)\right] + e^k_\gamma \left(\mathbf{d}e^\gamma_p - e^\gamma_{p\sigma}\mathbf{d}x^\sigma\right) \wedge \left[e^p_\sigma \left(\mathbf{d}e^\sigma_l - e^\sigma_{l\rho}\mathbf{d}x^\rho\right)\right]\\
&= \mathbf{d}e^k_\gamma \wedge \left(\mathbf{d}e^\gamma_l - e^\gamma_{l\rho}\mathbf{d}x^\rho\right) - e^k_\gamma \mathbf{d}e^\gamma_{l\rho} \wedge \mathbf{d}x^\rho +\\
&\qquad + e^k_\gamma e^p_\sigma \left[\mathbf{d}e^\gamma_p \wedge \mathbf{d}e^\sigma_l + \left(e^\gamma_{p\beta}\mathbf{d}e^\sigma_l \wedge \mathbf{d}x^\beta - e^\sigma_{l\beta}\mathbf{d}e^\gamma_p \wedge \mathbf{d}x^\beta\right) + e^\gamma_{p\beta}e^\sigma_{l\delta}\mathbf{d}x^\beta \wedge \mathbf{d}x^\delta\right]\\
&= -e^\gamma_{l\rho}\mathbf{d}e^k_\gamma \wedge \mathbf{d}x^\rho - e^k_\gamma \mathbf{d}e^\gamma_{l\rho} \wedge \mathbf{d}x^\rho +\\
&\qquad + e^k_\gamma e^p_\sigma \left[\left(e^\gamma_{p\beta}\mathbf{d}e^\sigma_l \wedge \mathbf{d}x^\beta - e^\sigma_{l\beta}\mathbf{d}e^\gamma_p \wedge \mathbf{d}x^\beta\right) + e^\gamma_{p\beta}e^\sigma_{l\delta}\mathbf{d}x^\beta \wedge \mathbf{d}x^\delta\right]
\end{aligned}$$

where in the passage from the third to the fourth line it was used the identity

$$\mathbf{d}e^k_\gamma \wedge \mathbf{d}e^\gamma_l + e^k_\gamma e^p_\sigma \mathbf{d}e^\gamma_p \wedge \mathbf{d}e^\sigma_l = 0.$$

Because of the identity

$$e^k_\gamma \mathbf{d} e^\gamma_p = -e^\gamma_p \mathbf{d} e^k_\gamma$$

we can reduce further the expression for Ω^k_l

$$\Omega^k_l = e^k_\gamma \left[-\mathbf{d} e^\gamma_{l\rho} \wedge \mathbf{d}x^\rho + e^p_\sigma \left(e^\gamma_{p\beta} \mathbf{d} e^\sigma_l \wedge \mathbf{d}x^\beta + e^\gamma_{p\beta} e^\sigma_{l\delta} \mathbf{d}x^\beta \wedge \mathbf{d}x^\delta \right) \right].$$

Take note that

$$e^l_\mu \Omega^k_l = e^k_\gamma \left(\mathbf{d}\Gamma^\gamma_{\mu\rho} \wedge \mathbf{d}x^\rho + \Gamma^\gamma_{\sigma\beta} \Gamma^\sigma_{\mu\delta} \mathbf{d}x^\beta \wedge \mathbf{d}x^\delta \right), \tag{3}$$

so that if we fix a connection Γ through its Christoffel symbols $(\Gamma^\mu_{\nu\sigma})$ in the canonical basis $\{\partial/\partial x^\mu\}$, then we will have that $e^\gamma_k = \delta^\gamma_k$ and this formula reduces to

$$\Omega^\mu_\nu := e^\nu_k e^l_\mu \Omega^k_l = \mathbf{d}\Gamma^\mu_{\nu\rho} \wedge \mathbf{d}x^\rho + \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\delta} \mathbf{d}x^\beta \wedge \mathbf{d}x^\delta$$

providing us with the usual formula for the connection in terms of the local coordinates.

Next we can provide a local expression for the map $\tilde{\sigma}_\Gamma : LM \to J^1 LM$. First we realize that a connection Γ is locally described by a map

$$\Gamma : x^\mu \mapsto \left(x^\mu, \Gamma^\sigma_{\mu\nu}(x) \right);$$

in these terms, the map $\tilde{\sigma}_\Gamma$ is given by

$$\tilde{\sigma}_\Gamma : \left(x^\mu, e^k_\nu \right) \mapsto \left(x^\mu, e^k_\nu, -e^\mu_k \Gamma^\sigma_{\mu\nu}(x) \right).$$

It is convenient to stress about an abuse of language committed here: We are indicating with the same symbol $\tilde{\sigma}_\Gamma$ either the map itself and its local version. Nevertheless, we obtain the following local expression for the connection form associated to Γ, namely

$$\left(\tilde{\sigma}^*_\Gamma \omega \right)^k_l = e^k_\mu \left(\mathbf{d} e^\mu_l + e^\sigma_l \Gamma^\mu_{\sigma\rho}(x) \, \mathbf{d}x^\rho \right).$$

In our approach this equation is equivalent to the so called *tetrad postulate*, which relates the components *of the same connection* in the two representations provided by the theory developed here: As a section Γ of the bundle of connections, and as an equivariant map $\tilde{\sigma}_\Gamma : LM \to J^1 LM$ such that the following diagram commutes

$$\begin{tikzcd} LM \arrow[r, "\tilde{\sigma}_\Gamma"] \arrow[d, "\tau"'] & J^1 LM \arrow[d, "p_{GL(n)}"] \\ M \arrow[r, "\Gamma"'] & C(LM) \end{tikzcd}$$

According to the previous discussion, the pullback of these forms along the section $s : x^\mu \mapsto (x^\mu, e^\nu_k(x))$ provides us with the expression for the connection forms associated to the underlying moving frame

$$e_k(x) := e^\nu_k(x) \frac{\partial}{\partial x^k};$$

in fact, given another such section $\bar{s} : x^\mu \mapsto (x^\mu, \bar{e}^\nu_k(x))$, there exists a map $g : x^\mu \mapsto \left(g^k_l(x) \right) \in GL(n)$ relating them, namely

$$\bar{e}^\mu_k(x) = g^l_k(x) \, e^\mu_l(x)$$

and so

$$\bar{s}^* \left(\tilde{\sigma}^*_\Gamma \omega \right)^k_l = h^k_p g^q_l s^* \left(\tilde{\sigma}^*_\Gamma \omega \right)^p_q + h^k_p \mathbf{d} g^p_l.$$

It allows us to answer the concerns raised in the introduction: The Palatini Lagrangian is a global form on $J^1 LM$, but this is false for its pullback along a local section. Namely, its global description needs the inclusion of information about the 1-jet of the vielbein involved in the local representation of the connection.

5. The Palatini gravity

5.1. **Degrees of freedom and restriction EDS for Palatini gravity.** A *tetrad* or, more generally, a *vielbein*, is a local isomorphism

$$e : TM \to M \times \mathbb{R}$$

or equivalently, a basis for the tangent bundle to M on an open set $U \subset M$. The rationale behind these objects is simply to replace the basis of the tangent bundle induced by the coordinates with a more general basis, perhaps determined by geometrical insights related to the formulation of the problem at hands. In fact, our approach to Palatini gravity is based in the replacing of the metric by an (by definition) orthonormal local frame; a change in the metric is thus performing by a change in the vielbein. Nevertheless, it is necessary to point out an essential difference between our approach and some of the descriptions of the Palatini gravity that can be found in the literature (see for example [BM94]), where it is assumed that the tangent bundle TM is isomorphic to $M \times \mathbb{R}^n$ (the "fake tangent bundle" viewpoint): We keep the local character of the frame bundle, using it only as a tool that permits us to describe a connection on M, without additional topological assumptions on this manifold, namely, by considering it as a parallelizable manifold.

From the mathematical viewpoint, a vielbein is nothing but a local section of the frame bundle LM; a connection on M, on the other side, can be considered as a section of the bundle of connections $C(LM)$. Because our fields are vielbeins and connections, the underlying bundle associated to the variational problem of Palatini gravity can be chosen as $C(LM) \times_M LM$, a bundle isomorphic to J^1LM. Nevertheless, it will be necessary to impose some restrictions on these connections, because it must be related to the metric implicitly described by the vielbein:

- **Metricity:** If the connection provides us with a covariant derivative, it will be desirable the preservation of the implicit metric, and a local section $s : M \to J^1LM$ will fulfill this requeriment if and only if
$$s^*\left(\eta^{ik}\omega_k^j + \eta^{jk}\omega_k^i\right) = 0,$$
where $\omega \in \Omega^1\left(J^1LM, \mathfrak{gl}(n)\right)$ is the canonical connection form on J^1LM and η is a Lorentzian fixed metric on $\mathbb{R}^n$ ($n = \dim M$, of course).
- **Zero torsion:** Additionally it will be required that
$$s^*T = 0$$
for $T \in \Omega^1\left(J^1LM, \mathbb{R}^n\right)$ the canonical form corresponding to the torsion.

These restrictions give rise to the restriction EDS for Palatini gravity. Thus, although the underlying bundle for this variational problem is a jet space, the restriction EDS is different from the contact structure; it is not totally unexpected, because the contact structure imposes on a local section the requeriment

$$s^*\omega = 0,$$

forcing the connection to be flat, a too strong condition for a vacuum gravitational field.

5.2. **The Lagrangian form of Palatini gravity.** Let us make use of these considerations in order to find a useful description of the Lagrangian form for Palatini gravity. Let us remember that our viewpoint is different from the classical approach: In the latter the field is the metric, namely, a section of the bundle $S(T^*M \otimes T^*M)$, where M is the spacetime and S stands for the symmetric part of the tensor product; in the former, we are taking as fields the sections of the bundle $p_1 : J^1LM \to M$. In order to formulate a Lagrangian on J^1LM we recall that there exists on this space a $\mathbb{R}^n$-valued 1-form θ, namely, the pullback along τ_{10} of the canonical form $\bar{\theta}$ on LM, and, additionaly, the $\mathfrak{gl}(n)$-valued 2-form Ω, just constructed as the curvature 2-form associated to the canonical connection on J^1LM induced by the contact structure. For every $k = 1, \cdots, n$ we can define the $\bigwedge^k(\mathbb{R}^n)$-valued k-form

$$\theta^{(k)}(X_1, \cdots, X_k) := \theta(X_1) \wedge \cdots \wedge \theta(X_k), \quad X_1, \cdots, X_k \in \mathfrak{X}\left(J^1LM\right)$$

and it allows us to define the $\bigwedge^2(\mathbb{R}^n)$-valued $n-2$-form Ξ via

$$\tilde{\Xi} := *\left(\theta^{(n-2)}\right),$$

where $*:\bigwedge^{n-i}(\mathbb{R}^n)\to\bigwedge^{i}(\mathbb{R}^n)$ is the Hodge star operator induced on the exterior algebra of $\mathbb{R}^n$ by η. Therefore we can use the antisymmetrization operator

$$A:\bigwedge^2(\mathbb{R}^n)\to\mathbb{R}^n\otimes(\mathbb{R}^n)^*=(\mathfrak{gl}(n))^*:u\wedge v\mapsto\frac{1}{2}\left[v\otimes\eta(u,\cdot)-u\otimes\eta(v,\cdot)\right]$$

in order to define a $(\mathfrak{gl}(n))^*$-valued $n-2$-form, namely

$$\Xi := A\left(\tilde{\Xi}\right);$$

the *Palatini Lagrangian* is

$$\lambda_{PG} := \langle \Xi \overset{\wedge}{,} \Omega\rangle \tag{4}$$

where $\langle\cdot\overset{\wedge}{,}\cdot\rangle:(\mathfrak{gl}(n))^*\otimes\mathfrak{gl}(n)\to\mathbb{R}$ indicates the extension of the contraction of a form with a vector to $\mathfrak{gl}(n)$ and $(\mathfrak{gl}(n))^*$-valued forms.

Note 15. The extension of operations from $\mathfrak{g}$ and $\mathbb{R}^n$ to $\mathfrak{g}$- and $\mathbb{R}^n$-valued forms is detailed in the work of Kôlar *et al.* [KMS+93], p. 100. In particular, the product structure on the $\mathbb{R}^n$-valued forms that yields to the form $\theta^{(n-2)}$ can be considered as an specialization of a more general structure found on vector valued forms.

5.3. **Some useful notation.** We can introduce some useful notation in order to handle with our formulas. Namely, if $\{\mathbf{e}_1,\cdots,\mathbf{e}_n\}$ is the canonical basis on $\mathbb{R}^n$, we can write $\theta:=\theta^i\mathbf{e}_i$ for some collection $\{\theta^i\}$ of 1-forms, and from here it can be concluded that

$$A\left(\theta^{(2)}\right)=\theta^i\wedge\theta^j\eta(\mathbf{e}_i,\cdot)\,\mathbf{e}_j.$$

As for the operator $*$, we can conclude with the formula

$$*\left(\theta^i\wedge\theta^j\right)=\eta^{ik}\eta^{jl}\theta_{kl},$$

where is was introduced the set of $n-p$-forms

$$\begin{aligned}\theta_{i_1\cdots i_p} &:= \frac{1}{(n-p)!}\epsilon_{i_1\cdots i_p i_{p+1}\cdots i_n}\theta^{i_{p+1}}\wedge\cdots\wedge\theta^{i_n}\\ &= X_{i_p}\lrcorner\cdots\lrcorner X_{i_1}\lrcorner\sigma_0;\end{aligned}$$

these forms are useful when dealing with the so called *Sparling forms*, see [DVM87]. Therefore for taking $\{\mathbf{e}^1,\cdots,\mathbf{e}^n\}$ as the dual basis of $\{\mathbf{e}_1,\cdots,\mathbf{e}_n\}$,

$$\begin{aligned}\Xi &:= \eta^{ik}\eta^{jl}\theta_{kl}\eta(\mathbf{e}_i,\cdot)\,\mathbf{e}_j\\ &= \eta^{jl}\theta_{kl}\mathbf{e}^k\otimes\mathbf{e}_j,\end{aligned}$$

and the Palatini Lagrangian can be written as

$$\lambda_{PG}=\eta^{kp}\theta_{kl}\wedge\Omega^l_p. \tag{5}$$

5.4. **The structure equations.** There are some equations that we need to take into account in this work. First we have the *structure equations*

$$\begin{aligned}\mathbf{d}\omega^i_j+\omega^i_k\wedge\omega^k_j &= \Omega^i_j\\ \mathbf{d}\theta^i+\omega^i_k\wedge\theta^k &= T^i,\end{aligned}$$

then its differential consequences, namely, the *Bianchi identities*

$$\begin{aligned}\mathbf{d}\Omega^i_j &= \Omega^i_k\wedge\omega^k_j-\omega^i_k\wedge\Omega^k_j\\ \mathbf{d}T^k &= \Omega^k_l\wedge\theta^l-\omega^k_l\wedge T^l,\end{aligned}$$

and some additional related identities

$$
\begin{aligned}
\mathbf{d}\theta_{li} &= \omega_l^k \wedge \theta_{ki} - \omega_i^k \wedge \theta_{kl} - \omega_s^s \wedge \theta_{li} + T^k \wedge \theta_{lik} \\
\mathbf{d}\Omega^{pq} &= \Omega_k^p \wedge \omega^{kq} - \omega_k^p \wedge \Omega^{kq} \\
\mathbf{d}\omega^{lp} &= -\omega_s^l \wedge \omega^{sp} + \Omega^{lp} \\
\mathbf{d}\theta_{ipq} &= \omega_i^k \wedge \theta_{kpq} + \omega_p^k \wedge \theta_{kqi} + \omega_q^k \wedge \theta_{kip} - \omega_s^s \wedge \theta_{ipq} + T^k \wedge \theta_{ipqk}
\end{aligned}
$$

where it were introduced the handy notations $\omega^{ij} := \eta^{jp}\omega_p^i, \Omega^{ij} := \eta^{jp}\Omega_p^i$.

6. Dynamics of the Palatini gravity

After this rather lenghty warm-up, we are ready to describe now Palatini gravity as a variational problem in the sense adopted in this work; as variational triple for this theory, as indicated above, we will take the bundle $J^1LM \to M$ as the underlying bundle; the Lagrangian form on this space will be λ_{PG}, defined in Eq. (4). Finally, the EDS restricting properly the sections of J^1LM is the one generated by metricity and torsionless conditions, namely

$$\mathcal{I}_{PG} = \langle \pi_{\mathfrak{p}}\omega, T \rangle_{\text{diff}};$$

here $\pi_{\mathfrak{p}} : \mathfrak{gl}(n) \to \mathfrak{p}$ is the projection onto the second summand in the Cartan decomposition $\mathfrak{gl}(n) = \mathfrak{so}(1, n-1) \oplus \mathfrak{p}$ induced by η.

It is necessary to point out that considerations of variational problems on the frame bundle, although from a slighty different point of view, can be found in the literature [BK04].

6.1. Considerations about admissible variations.

Given the existence of a restriction EDS, the variations to be considered in order to find out the equations of motion of Palatini gravity cannot be arbitrary; rather they must be restricted in some way. Let us recall that a *variation* of a section $s : M \to E$ of a bundle $E \to M$ is a section of the pullback bundle $s^*(VE)$, perhaps with compact support, and that the relevant variations for a variational problem are the infinitesimal symmetries of the restriction EDS. We could introduce the following definition in order to work here with these objects.

Definition 16 (Admissible variations)**.** *An* admissible variation *of the integral section s for an EDS $\mathcal{I}$ is a variation δs with an extension $\widehat{\delta s} \in \mathfrak{X}(E)$ which is an infinitesimal symmetry of $\mathcal{I}$, that is, such that*

$$s^*\left(\mathcal{L}_{\widehat{\delta s}}\mathcal{I}\right) = 0.$$

An admissible variation for an EDS $\mathcal{I}$ produces a path in the set of integral sections of this EDS. In terms of adapted coordinates $\left(x^\mu, e_k^\nu, e_{k\rho}^\sigma\right)$ on J^1LM, any variation reads

$$x^\mu \mapsto \left(0, \delta e_k^\nu, \delta e_{k\rho}^\sigma\right);$$

it means in particular that the canonical forms θ and ω can be varied independently. This freedom will be use in order to simplify the calculations below.

6.2. **The variations of the connection.** On the same open set U_α and using the previous identifications, we can consider that variations of the connection as $\mathfrak{gl}(n)$-valued 1-forms $\delta\omega^{ij}$. Therefore

$$\begin{aligned}
\delta_\omega \lambda_{PG} &= \theta_{ik} \wedge \left[\mathbf{d}\left(\delta\omega^{ik}\right) + \eta_{pq}\delta\omega^{pi} \wedge \omega^{kq} + \eta_{pq}\omega^{pi} \wedge \delta\omega^{kq}\right] \\
&= (-1)^{n+1}\, \mathbf{d}\theta_{ik} \wedge \delta\omega^{ik} + \theta_{ik} \wedge \left(-\eta_{pq}\omega^{kq} \wedge \delta\omega^{pi} + \eta_{pq}\omega^{pi} \wedge \delta\omega^{kq}\right) \\
&= (-1)^{n+1}\left(\eta_{ip}\omega^{lp} \wedge \theta_{lk} - \eta_{kp}\omega^{lp} \wedge \theta_{li} + T^l \wedge \theta_{ikl}\right) \wedge \delta\omega^{ik} + \\
&\qquad\qquad + \theta_{ik} \wedge \left(-\eta_{pq}\omega^{kq} \wedge \delta\omega^{pi} + \eta_{pq}\omega^{pi} \wedge \delta\omega^{kq}\right) \\
&= \left[(-1)^{n+1}\left(\eta_{ip}\omega^{lp} \wedge \theta_{lk} - \eta_{kp}\omega^{lp} \wedge \theta_{li} + T^l \wedge \theta_{ikl}\right) - \eta_{iq}\theta_{kl} \wedge \omega^{lq} + \eta_{pk}\theta_{li} \wedge \omega^{pl}\right] \wedge \delta\omega^{ik} \\
&= \left[-\eta_{ip}\theta_{lk} \wedge \omega^{lp} + \eta_{kp}\theta_{li} \wedge \omega^{lp} + (-1)^{n+1}\, T^l \wedge \theta_{ikl} - \eta_{iq}\theta_{kl} \wedge \omega^{lq} + \eta_{pk}\theta_{li} \wedge \omega^{pl}\right] \wedge \delta\omega^{ik} \\
&= \left[\eta_{kp}\theta_{li} \wedge \left(\omega^{lp} + \omega^{pl}\right) + (-1)^{n+1}\, T^l \wedge \theta_{ikl}\right] \wedge \delta\omega^{ik}.
\end{aligned} \tag{6}$$

Therefore the variations of the Lagrangian λ_{PG} annihilates, independently of the form of the variations $\delta\omega$, and so its does not contribute to the equations of motion.

6.3. **Considerations about the variations of the frame.** It is time to see what the variations of the frame produce on the $n-2$-forms θ_{ij} defined previously. We will consider here variations with its support on a chart U_α. Now,

$$\mathcal{L}_{\delta s}\theta_{ij} = (-1)^{i+j+1}\left[\delta\theta^1 \wedge \cdots \wedge \widehat{\theta^i} \wedge \cdots \wedge \widehat{\theta^j} \wedge \cdots \wedge \theta^n + \cdots \right.$$
$$\left. \cdots + \theta^1 \wedge \cdots \wedge \widehat{\theta^i} \wedge \cdots \wedge \widehat{\theta^j} \wedge \cdots \wedge \delta\theta^n\right].$$

Therefore

$$\mathcal{L}_{\delta s}\theta_{ij} = \delta\theta^k \wedge \theta_{kij}. \tag{7}$$

So by performing the variations of the frame, we obtain

$$\delta_{\xi_1}\lambda_{PG} = \delta\theta^m \wedge \theta_{mki} \wedge \left(\mathbf{d}\omega^{ki} + \eta_{lm}\omega^{li} \wedge \omega^{km}\right),$$

namely

$$\theta_{jki} \wedge \left(\mathbf{d}\omega^{ki} + \eta_{lm}\omega^{li} \wedge \omega^{km}\right) = 0. \tag{8}$$

As an additional formula useful in dealing with the variations of the frame, we can calculate the differential of the forms θ_{ij}, expressing them in terms of the connection and the associated torsion. Namely, by using the definition

$$\mathbf{d}\theta^i = -\omega^i_k \wedge \theta^k + T^i$$

we will obtain that

$$\mathbf{d}\theta_{ij} = \omega^k_i \wedge \theta_{kj} - \omega^k_j \wedge \theta_{ki} - \omega^k_k \wedge \theta_{ij} + T^k \wedge \theta_{ijk}$$

where, as above

$$\theta_{ijk} := X_k \lrcorner X_j \lrcorner X_i \lrcorner \sigma_0.$$

As shown in [Thi86], the equations of motion (8) are equivalent to the annihilation of the Einstein tensor.

6.4. **Discussion: The global form for Einstein equations and the underlying metric.** According to the previous calculations, the equations of motion for the Palatini gravity can be described as the EDS generated by the forms

$$\begin{cases}\mathbf{d}\theta^i+\omega^i_k\wedge\theta^k,\\ \theta_{ipq}\wedge\Omega^{pq},\\ \eta^{ip}\omega^j_p+\eta^{jp}\omega^i_p.\end{cases}\tag{9}$$

It is interesting to note that these expressions are global; we can think on them as a global form for vacuum Einstein equation. Additionally the jet space J^1LM has a $GL\left(n\right)$-action, obtained by lifting the corresponding action of $GL\left(n\right)$ to the frame bundle; in terms of the adapted coordinates, it reads

$$g\cdot\left(x^\mu,e^\nu_k,e^\sigma_{k\rho}\right)=\left(x^\mu,g^l_ke^\nu_l,g^l_ke^\sigma_{l\rho}\right).$$

This action is involved in the proof of the next proposition, giving sense to our choice of the relevant fields for describing gravity.

Proposition 17. *Let $s,\bar{s}:U\to J^1LM$ be a pair of solutions for the problem posed by* (9). *Then there exists an smooth map $g:U\to SO\left(1,n-1\right)$ such that $\bar{s}=g\cdot s$.*

Proof. If $s:U\to J^1LM$ is a local solution for these equations and $k:U\to SO\left(1,n-1\right)$ is an smooth map, we will have that $s':=k\cdot s$ verifies

$$s'^*\left(\mathbf{d}\theta^i+\omega^i_k\wedge\theta^k\right)=0=s'^*\left(\theta_{ipq}\wedge\Omega^{pq}\right)$$

and $s'^*\left(\eta^{ip}\omega^j_p+\eta^{jp}\omega^i_p\right)=0$. Now for $s,\bar{s}$ there exists $h:U\to GL\left(n\right)$ such that $\bar{s}=g\cdot s$; by using a Cartan decomposition of $GL\left(n\right)$ respect to the form η we can factorize $GL\left(n\right)=P\cdot SO\left(1,n-1\right)$ where P is the set of η-symmetric matrices, and if $h=p\cdot k$, we see that the η-symmetric factor p must verify

$$\mathrm{Ad}_{p^{-1}}\omega+p^{-1}\mathbf{d}p\in\mathfrak{so}(1,n-1)$$

for $\omega\in\mathfrak{so}(1,n-1)$. But there exists $a:U\to SO\left(1,n-1\right)$ such that $p=ada^{-1}$, where $d:U\to P$ is a diagonal matrix; therefore the previous requeriment on p translates into

$$\mathrm{Ad}_{d^{-1}}\tilde{\omega}+d^{-1}\mathbf{d}d\in\mathfrak{so}(1,n-1)$$

with $\tilde{\omega}\in\mathfrak{so}(1,n-1)$. But the first summand in this expression has zero entries in the diagonal, and the second is a diagonal matrix, so it can be split as the pair of conditions

$$\mathrm{Ad}_{d^{-1}}\tilde{\omega}\in\mathfrak{so}(1,n-1),\qquad d^{-1}\mathbf{d}d=0$$

for $\tilde{\omega}\in\mathfrak{so}(1,n-1)$; this means that d must be locally constant, and the first forces $d=\mathrm{Id}$. □

Therefore the local solutions determine a unique metric h according to the formula

$$h:=\eta_{kl}e^k_\mu e^l_\nu\mathbf{d}x^\mu\otimes\mathbf{d}x^\nu.$$

The first and third forms in the above EDS are enough to determine uniquely the connection, or more precisely the functions $e^\mu_{k\nu}$ of a solution, from the frame functions e^μ_k. This result follows at once by using Proposition 29; as we will see below, these functions determines a connection on M that is the Levi-Civita connection for h. Therefore the second set of forms are the true equations of motion for the metric.

6.5. **Differential consequences of the vacuum Einstein equations.** We will use the structure equations and its differential consequences in order to find a set of algebraic generators for the EDS

$$\begin{aligned}\mathcal{I}_{\mathrm{E}}&:=\left\langle\eta_{kp}\theta_{li}\wedge\left(\omega^{lp}+\omega^{pl}\right)+(-1)^{n+1}T^l\wedge\theta_{ikl},\Omega^{pq}\wedge\theta_{ipq}\right\rangle_{\mathrm{diff}}\\&=\left\langle\omega^{lp}+\omega^{pl},T^l,\Omega^{pq}\wedge\theta_{ipq}\right\rangle_{\mathrm{diff}}.\end{aligned}$$

The differential of the first set of generators $\omega^{lp}+\omega^{pl}$ yields to

$$\begin{aligned}\mathbf{d}\left(\omega^{lp}+\omega^{pl}\right)&=-\eta_{st}\left(\omega^{lt}\wedge\omega^{sp}+\omega^{ps}\wedge\omega^{tl}\right)+\left(\Omega^{lp}+\Omega^{pl}\right)\\&=-\eta_{st}\left[\left(\omega^{lt}+\omega^{tl}\right)\wedge\omega^{sp}-\omega^{tl}\wedge\left(\omega^{sp}+\omega^{ps}\right)\right]+\left(\Omega^{lp}+\Omega^{pl}\right),\end{aligned}$$

so the antisymmetry property for the curvature

$$A^{lp}:=\Omega^{lp}+\Omega^{pl}=0$$

is a differential consequence of the original Einstein equations. From the second Bianchi identity

$$\mathbf{d}T^k=\Omega^k_l\wedge\theta^l-\omega^k_l\wedge T^l$$

another generator for the EDS $\mathcal{I}_E$ is $B^k:=\Omega^k_l\wedge\theta^l$. Finally, from the last set of generators we obtain the differential consequences

$$\mathbf{d}\left(\Omega^{pq}\wedge\theta_{ipq}\right)=\eta_{kl}\Omega^{pk}\wedge\left(\omega^{lq}+\omega^{ql}\right)\wedge\theta_{ipq}+\left(\omega^k_i-\delta^k_i\omega^s_s\right)\wedge\Omega^{pq}\wedge\theta_{kpq}+\Omega^{pq}\wedge T^k\wedge\theta_{ipqk};$$

therefore there are no new algebraic generators from here. In conclusion

$$\mathcal{I}_E=\left\langle\omega^{lp}+\omega^{pl},\Omega^{lp}+\Omega^{pl},\mathbf{d}\theta^l+\omega^l_k\wedge\theta^k,\Omega^k_l\wedge\theta^l,\Omega^{pq}\wedge\theta_{ipq}\right\rangle_{\text{alg}}$$

is a presentation for $\mathcal{I}_E$ in terms of algebraic generators.

7. Reduction for a variational problem

An important observation concerning the variational problem

$$\left(J^1LM\to M,\lambda_{PG},\mathcal{I}_{PG}\right)$$

is that both λ_{PG} and $\mathcal{I}_{PG}$ are $SO\left(1,n-1\right)$-invariant. It will be interesting to find a procedure in order to quotient out the degrees of freedom associated to the orbits of this symmetry group, namely, if we can apply a kind of reduction procedure, as in [LR03]. The problem with this approach is that in this reference the authors deal with reduction of the so called *classical variational problem*, namely, with variational problems of the form

$$\left(J^1P\to M,\mathcal{L}\omega,\mathcal{I}_{\text{con}}\right),$$

where $P\to M$ is a principal bundle, $\mathcal{L}\in C^\infty\left(J^1P\right)$, ω is a volume form on M and $\mathcal{I}_{\text{con}}$ is the contact structure on J^1P. Therefore we must devise a reduction scheme general enough to include variational problems whose restriction EDS are different from the contact EDS of a jet space.

7.1. Reduction of an EDS.

In order to set the reduction procedure for a variational problem, it is crucial to know how to reduce the restriction EDS. So let M be a manifold, G a Lie group acting on M in such a way that the space of orbits $\overline{M}:=M/G$ is a manifold; we will denote by $p_G:M\to\overline{M}$ the canonical projection. Let us consider $\mathcal{I}$ an EDS on M such that

$$g\cdot\mathcal{I}\subset\mathcal{I}\qquad\forall g\in G;$$

the following definition can be found in [AF05].

Definition 18 (Reduced EDS)**.** *The* reduced EDS *associated to the action of G on $(M,\mathcal{I})$ is the set of forms*

$$\overline{\mathcal{I}}:=\left\{\alpha\in\Omega^\bullet\left(\overline{M}\right):p_G^*\alpha\in\mathcal{I}\right\}.$$

Let $\tau:P\to M$ be a G-principal bundle. This definition can be applied in order to reduce the contact structure on J^1P: It will give us an interpretation of the canonical 2-form Ω_2 on $C\left(P\right)$ as generator of the EDS on the bundle of connection obtained by reduction of the contact structure of J^1P, as the following example shows.

Example 19 (Reduction of the contact structure on J^1P)**.** Let us analyze this in more detail; the result we are looking for is local, so there is no real loss in assuming that $P = M \times G$, and this means that $J^1P = P \times_M (T^*M \otimes \mathfrak{g})$, by using the following correspondence: If $s : M \to P$ is a section, then

$$j^1_x s = \left(x, s(x), (T_x s)(\cdot)(s(x))^{-1}\right)$$

for all $x \in M$. It means that $C(P) = T^*M \otimes \mathfrak{g}$, and the canonical projection $q : J^1P \to C(P)$ is simply

$$q(x, g, \xi) = (x, \xi).$$

In these terms the 2-form Ω_2 reads

$$\Omega_2|_{(x,\xi)} = \mathbf{d}\xi - \frac{1}{2}[\xi \stackrel{\wedge}{,} \xi].$$

The contact structure is generated by the 1-forms

$$\theta|_{(x,g,\xi)} := \mathbf{d}g \cdot g^{-1} - \xi;$$

the G-action on P is $h \cdot (x, g) = (x, gh)$, that lifts to $h \cdot (x, g, \xi) = (x, gh, \xi)$. Therefore a set of algebraic generators for $\mathcal{I}$ is in this case

$$\mathcal{G} := \{\theta, \mathbf{d}\theta\} = \left\{\mathbf{d}g \cdot g^{-1} - \xi, (1/2)[\xi \stackrel{\wedge}{,} \xi] - \mathbf{d}\xi\right\};$$

bearing in mind future applications, the 2-degree generator $\mathbf{d}\theta$ has been written in a convenient form. Thus we have that $J^1P/G = T^*M \otimes \mathfrak{g}$ with projection given by

$$p_G(x, g, \xi) = (x, \xi).$$

The quotient EDS $\bar{\mathcal{I}}$ is graded, as $\mathcal{I}$ does and p_G^* is a 0-degree morphism; if α is a 1-form in $\bar{\mathcal{I}}$, we will have that

$$p_G^*\alpha = f \cdot \theta$$

for some $f \in C^\infty\left(J^1P\right)$, and then if $(0, \zeta, 0) \in T_{(x,g,\xi)}J^1P$, it results that

$$0 = (p_G^*\alpha)(0, \zeta, 0) = f(x, g, \xi)\,\zeta.$$

So $p_G^*\alpha = 0$ and it means that $\alpha = 0$, from the fact that p_G is a surjective map. If $\beta \in \bar{\mathcal{I}} \cap \Omega^p\left(J^1P/G\right), p > 1$, we will have that

$$p_G^*\beta = \mu \wedge \theta + \nu \wedge ((1/2)[\xi \stackrel{\wedge}{,} \xi] - \mathbf{d}\xi)$$

for some $\mu \in \Omega^{p-1}\left(J^1P/G\right)$ and $\nu \in \Omega^{p-2}\left(J^1P/G\right)$; by performing the replacement

$$\mathbf{d}g \cdot g^{-1} = \xi + \theta$$

we can assume that neither μ nor ν have dependence in the g-variable. Therefore, by contracting this identity with an infinitesimal generator for the G-action, we obtain that

$$\begin{aligned} 0 &= (0, \zeta, 0) \lrcorner (p_G^*\beta) \\ &= (-1)^{p+1}\zeta\mu \end{aligned}$$

and so $\mu = 0$; from here we can conclude that

$$\bar{\mathcal{I}} = \left\langle \frac{1}{2}[\xi \stackrel{\wedge}{,} \xi] - \mathbf{d}\xi \right\rangle_{\text{alg}},$$

meaning that the reduced EDS is generated by Ω_2.

The previous example gives some insight in the subtleties concerning the reduction of an EDS: The original contact structure is locally generated by 1-forms, but the reduced EDS is generated by a collection of 2-forms. Nevertheless, there exists a result allowing us to find a set of generators for a reduced EDS, under mild conditions, namely, by requiring the generators to be pullback of some forms along a projection. It is convenient to note that it was not fulfilled in the previous example, because $\mathbf{d}g \cdot g^{-1} - \xi$ is not the pullback along p_G of any form on $M \times \mathfrak{g}$.

Proposition 20. *Let $p: M \to N$ be a fibration and $\mathcal{I} := \langle \alpha_1, \cdots, \alpha_p \rangle_{diff}$ a differential ideal such that $\alpha_i \in \Omega^{k_i}(M)$ for some integers k_i. Let us suppose that on N there exists a set of forms $\{\beta_1, \cdots, \beta_p\}$ such that*

$$\alpha_i = p^* \beta_i \qquad \text{for } i = 1, \cdots, p.$$

Then $\overline{\mathcal{I}} = \langle \beta_1, \cdots, \beta_p \rangle_{diff}$.

Proof. The inclusion $\langle \beta_1, \cdots, \beta_p \rangle_{\text{diff}} \subset \overline{\mathcal{I}}$ follows from the definition of reduced EDS. On the other side, if $\omega \in \overline{\mathcal{I}}$, there exists $\gamma_1, \cdots, \gamma_p$ such that

$$p^* \omega = \gamma_1 \wedge p^* \beta_1 + \cdots + \gamma_p \wedge p^* \beta_p.$$

Thus it is enough to prove that it implies $\gamma_i = p^* \sigma_i$ for all $i = 1, \cdots, p$. □

Definition 21 (Reduction of a variational problem)**.** *Let $(\Lambda, \lambda, \mathcal{I})$ be a variational problem on the bundle $p: \Lambda \to M$. Let us suppose that a Lie group G acts on Λ such that*

1. *the action is free and proper,*
2. *its orbits are vertical, i.e. $p(g \cdot u) = p(u)$ for all $u \in \Lambda$ and $g \in G$,*
3. *there exists $\bar{\lambda} \in \Omega^n(\bar{M})$ such that $p_G^* \bar{\lambda} = \lambda$, where $p_G : \Lambda \to \overline{\Lambda} := \Lambda/G$ is the canonical projection, and*
4. *it is a symmetry group for the EDS $\mathcal{I}$.*

The reduced variational problem *for $(\Lambda, \lambda, \mathcal{I})$ is the variational problem $\left(\overline{\Lambda}, \overline{\lambda}, \overline{\mathcal{I}}\right)$, where $\overline{\mathcal{I}}$ is the reduced EDS for $\mathcal{I}$.*

Example 22 (Euler-Poincaré reduction)**.** The Euler-Poincaré reduction [LRS00, CL12] can be seen as an instance of this reduction scheme. In the setting of Example 19, we see that the reduced variational problem associated to $\left(J^1 P, L\omega, \mathcal{I}_{\text{con}}\right)$, where $L \in C^\infty\left(J^1 P\right)^G$ and ω is an invariant volume, is nothing but $\left(C(P), \overline{L}\omega, \langle \Omega_2 \rangle_{\text{diff}}\right)$. The relationship with Euler-Poincaré reduction can be revealed by means of the following consideration: The restrictions on the possible variations of the fields of the reduced field theory are exactly those defining the allowed variations (see Definition 16), namely, the infinitesimal variations for the restriction EDS. If $\sigma : U \subset M \to C(P) : x \mapsto (x, \xi(x))$ is a local section of the bundle of connections, integral for $\langle \Omega_2 \rangle_{\text{diff}}$, then a vertical vector field $(0, \Xi)$ will be an infinitesimal symmetry of σ iff

$$\sigma^* \left(\mathbf{d}\Xi - [\Xi, \xi]\right) = 0.$$

Let $\text{ad}(P) := (P \times \mathfrak{g})/G$ be the adjoint bundle associated to P. By using the fact that $C(P)$ is an affine bundle modelled on $T^*M \otimes \text{ad}(P)$, we can identify any variation Ξ of its sections as an $\text{ad}(P)$-valued 1-form on M. In these terms the requeriment of admissibility for variations reads $\mathbf{d}_\sigma \Xi = 0$, where $\mathbf{d}_\sigma$ is the covariant exterior differential associated to the connection σ; by using that σ is flat, then (at least locally) there exists a section $\eta : U \subset M \to \text{ad}(P)$ for the adjoint bundle such that

$$\Xi = \mathbf{d}_\sigma \eta.$$

For H an arbitrary connection, we see from here that

$$\Xi = \mathbf{d}_{H+\sigma-H} \eta = \mathbf{d}_H \eta + [\sigma - H, \eta],$$

the usual requeriment for variations in Euler-Poincaré reduction (compare with Prop. 3.1 in [LRS00]).

7.2. **Reduced Palatini gravity.** We are ready to perform the reduction of the variational problem $\left(J^1 LM, \lambda_{PG}, \mathcal{I}_{PG}\right)$ by the Lorentz subgroup $H := SO(1, n-1)$. The H-action is readily seen to be free and proper, and its orbits are vertical, so it remains to verify that λ_{PG} and $\mathcal{I}_{PG}$ are H-invariant.

In order to properly show this invariance, we need to introduce a nice description for the quotient bundle

$$\begin{aligned} \tau : J^1 LM/H &\to M \\ \left[j^1_x s\right]_H &\mapsto x. \end{aligned}$$

Now let us remember that the bundle $J^1LM \to J^1LM/GL(n)$ is isomorphic to the pullback bundle p^*J^1LM, where $p : J^1LM/GL(n) \to M$ is the projection induced by $\tau_1 : J^1LM \to M$; from this perspective every element $j^1_x s$ of J^1LM can be written as a pair

$$j^1_x s = \left(\left[j^1_x s \right]_{GL(n)}, s(x) \right),$$

and the $GL(n)$-action is simply

$$\left(\left[j^1_x s \right]_{GL(n)}, s(x) \right) \cdot g = \left(\left[j^1_x s \right]_{GL(n)}, s(x) \cdot g \right).$$

Then we have the following representation for the quotient J^1LM/H.

Lemma 23. *Let $[\tau] : \Sigma \to M$ be the bundle of metrics on M. Then J^1LM/H is isomorphic to the pullback bundle $p^*\Sigma = C(LM) \times_M \Sigma$.*

The local version for these results is very illuminating of the geometrical meaning of the sections of these bundles.

Proposition 24. *Let $\left(x^\mu, e^\nu_k, e^\sigma_{j\mu}\right)$ be the set of jet coordinates introduced above on J^1LM. Then there exists a set of coordinates $\left(x^\mu, g^{\mu\nu}, \Gamma^\sigma_{\rho\gamma}\right)$ on $p^*\Sigma$ such that*

$$\tilde{g}^{\mu\nu} := g^{\mu\nu} \circ p_H = \eta^{kl} e^\mu_k e^\nu_l, \qquad \tilde{\Gamma}^\sigma_{\rho\gamma} := \Gamma^\sigma_{\rho\gamma} \circ p_H = -e^\sigma_{k\gamma} e^k_\rho.$$

In terms of these functions

$$\lambda_{PG} = \epsilon_{\mu_1 \cdots \mu_{n-2}\gamma\kappa} \sqrt{-\det \tilde{g}}\, \tilde{g}^{\kappa\phi} \boldsymbol{d}x^{\mu_1} \wedge \cdots \wedge \boldsymbol{d}x^{\mu_{n-2}} \wedge \left(\boldsymbol{d}\tilde{\Gamma}^\gamma_{\rho\phi} \wedge \boldsymbol{d}x^\rho + \tilde{\Gamma}^\sigma_{\delta\phi} \tilde{\Gamma}^\gamma_{\beta\sigma} \boldsymbol{d}x^\beta \wedge \boldsymbol{d}x^\delta \right) \tag{10}$$

and

$$\eta^{ik}\omega^j_k + \eta^{jk}\omega^i_k = e^i_\mu e^j_\nu \left(\boldsymbol{d}\tilde{g}^{\mu\nu} + \left(\tilde{g}^{\mu\sigma} \tilde{\Gamma}^\nu_{\gamma\sigma} + \tilde{g}^{\nu\sigma} \tilde{\Gamma}^\mu_{\gamma\sigma} \right) \boldsymbol{d}x^\gamma \right) \tag{11}$$

$$T^i = e^i_\sigma \tilde{\Gamma}^\sigma_{\mu\nu} \boldsymbol{d}x^\mu \wedge \boldsymbol{d}x^\nu. \tag{12}$$

In particular, the Lagrangian form λ_{PG} is horizontal for the H-projection, and the EDS $\mathcal{I}_{PG}$ is H-invariant.

By combining these equations and Proposition 20 we deduce the following corollary.

Corollary 25. *The reduced Lagrangian $\overline{\lambda}_{PG}$ is given by*

$$\overline{\lambda_{PG}} = \epsilon_{\mu_1 \cdots \mu_{n-2}\gamma\kappa} \sqrt{-\det g}\, g^{\kappa\phi} \boldsymbol{d}x^{\mu_1} \wedge \cdots \wedge \boldsymbol{d}x^{\mu_{n-2}} \wedge \left(\boldsymbol{d}\Gamma^\gamma_{\rho\phi} \wedge \boldsymbol{d}x^\rho + \Gamma^\sigma_{\delta\phi} \Gamma^\gamma_{\beta\sigma} \boldsymbol{d}x^\beta \wedge \boldsymbol{d}x^\delta \right),$$

and the reduced EDS $\overline{\mathcal{I}}_{PG}$ can be generated as

$$\overline{\mathcal{I}}_{PG} = \left\langle \boldsymbol{d}g^{\mu\nu} + \left(g^{\mu\sigma}\Gamma^\nu_{\gamma\sigma} + g^{\nu\sigma}\Gamma^\mu_{\gamma\sigma} \right) \boldsymbol{d}x^\gamma, \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu} \right\rangle_{diff}.$$

Note 26 (Levi-Civita EDS). By recalling that J^1LM/H is the product bundle $C(LM) \times_M \Sigma$, the reduced EDS $\overline{\mathcal{I}}_{PG}$ can be interpreted geometrically: For every metric $g : M \to \Sigma$ on M, the unique connection $\Gamma : M \to C(LM)$ such that $\Gamma \times g : M \to C(LM) \times_M \Sigma$ is an integral section for $\overline{\mathcal{I}}_{PG}$ is the Levi-Civita connection for g. Thus we can call this EDS the *Levi-Civita EDS*.

Note 27 ($GL(4)$-invariant gravity). This setting provides us with enough tools in order to describe some other approaches to "gravity with moving frames". For example, we can set a variational problem for the so called $GL(4)$*-invariant gravity* [FP90a, FP90b]: In brief, in this theory the fields are a soldering form $\theta \in \mathrm{Iso}(TM, TM)$, a metric κ and a connection Γ on M. Using the previous identifications, we can consider these fields as sections of the bundles $(LM \times_M LM)/GL(n)$ (where $GL(n)$ is acting diagonally), Σ and $C(LM)$ respectively. By using that the metric κ is induced locally by a section e of LM, and that such section induces a local section

$$\tilde{e} : (LM \times_M LM)/GL(n) \to LM \times_M LM$$

via

$$[f_1, f_2]_{GL(n)} \mapsto (e(x), g \cdot f_2) \quad \text{iff } x := p(f_1) \text{ and } e(x) = g \cdot f_1,$$

the bundle $LM\times_M LM$ can be used instead of the first two bundles mentioned above; it amounts to describe the morphism θ by the way it is acting on a particular basis of TM, namely, the basis used in the description of the metric κ. The underlying bundle of the variational problem describing this kind of gravity theory will be $C(LM)\times_M LM\times_M LM = J^1LM\times_M LM$, which can be considered as a submanifold of $J^1LM\times_M J^1LM$ via the inclusion

$$\imath : C(LM)\times_M LM\times_M LM \hookrightarrow J^1LM\times_M J^1LM : (\Gamma, e, f)\mapsto (\Gamma, e;\Gamma, f).$$

By denoting $p_A, A=1,2$ the projections onto the first and second factor in $J^1LM\times_M J^1LM$, the restriction EDS is generated as follows

$$\mathcal{I}_{FP} := \left\langle \imath^*p_1^*\left(\eta^{ik}\omega_k^j+\eta^{jk}\omega_k^i\right), \imath^*p_2^*T\right\rangle_{\text{diff}};$$

these restrictions are nothing but Eqs. (2.1) and (2.2) in [FP90a]. Finally the Lagrangian considered by these authors is the pullback along p_2 of the Palatini Lagrangian defined above λ_{PG}, namely

$$\lambda_{FP} := p_2^*\lambda_{PG}.$$

Thus the variational problem for this version of gravity is the triple

$$\left(C(LM)\times_M LM\times_M LM, \lambda_{FP}, \mathcal{I}_{FP}\right).$$

7.3. **Discussion: Einstein gravity as a reduced variational problem for Palatini gravity.** The reduced Palatini variational problem can be considered as equivalent to the Einstein-Hilbert variational problem. In fact, we can consider the following diagram

$$\begin{tikzcd} & C(LM)\times_M J^1\Sigma \arrow[dl, "p_2"'] \arrow[dd] \arrow[dr, "\Pi"] & \\ J^1\Sigma \arrow[dr] & & C(LM)\times_M \Sigma \arrow[dl] \\ & M & \end{tikzcd}$$

induced by $J^1\Sigma\to\Sigma$; let us define

$$\mathcal{J} := \left\langle p_2^*\mathcal{I}^\Sigma_{\text{con}}, \Pi^*\overline{\mathcal{I}}_{PG}\right\rangle_{\text{diff}},$$

where $\mathcal{I}^\Sigma_{\text{con}}$ is the contact structure on $J^1\Sigma$. Then the next result follows.

Lemma 28. *There exists a manifold $L\subset C(LM)\times_M J^1\Sigma$ minimal with respect to the property that every integral manifold of $\mathcal{J}$ must be included in it. The map $p_2|_L : L\to J^1\Sigma$ is a bundle isomorphism such that $p_2^*\mathcal{I}^\Sigma_{con}=\mathcal{J}|_L$.*

Then the $\Pi-$projectable extremals of

$$\left(L, \Pi^*\overline{\lambda}_{PG}\big|_L, \mathcal{J}|_L\right)$$

are in one-to-one correspondence with the extremals of the reduced Palatini variational problem on $C(LM)\times_M\Sigma$ via Π, and with the extremals of the classical variational problem $\left(J^1\Sigma, \lambda_{EH}, \mathcal{I}^\Sigma_{\text{con}}\right)$ through $p_2|_L$; the Einstein-Hilbert Lagrangian λ_{EH} is determined by the equation

$$p_2^*\lambda_{EH} = \Pi^*\overline{\lambda}_{PG}\big|_L.$$

It induces the equivalence we were looking for.

8. Conclusions

In this work a geometrical formulation for Palatini gravity was provided, by using a broader notion for the term *variational problem.* In order to perform this task, it was necessary to use some constructions associated to the jet space of the frame bundle. This picture would give us some insights on the geometrical character of vacuum GR, complementary to those found in the literature. In order to relate this formulation with the usual Einstein-Hilbert variational problem, a generalized reduction scheme was set.

Appendix A. An important algebraic result

We would like to state here the following algebraic proposition.

Proposition 29. *Let* $\{c_{ijk}\}$ *be a set of real numbers such that*

$$\begin{cases} c_{ijk} \mp c_{jik} = b_{ijk} \\ c_{ijk} \pm c_{ikj} = a_{ijk} \end{cases}$$

for some given set of real numbers $\{a_{ijk}\}$ *and* $\{b_{ijk}\}$ *such that* $b_{ijk} \mp b_{jik} = 0$ *and* $a_{ijk} \pm a_{ikj} = 0$. *Then*

$$c_{ijk} = \frac{1}{2}\left(a_{ijk} + a_{jki} - a_{kij} + b_{ijk} + b_{kij} - b_{jki}\right)$$

is the unique solution for this linear system.

Proof. From first equation we see that

$$\pm c_{jik} = c_{ijk} - b_{ijk}.$$

The trick now is to form the following combination

$$\begin{aligned} a_{ijk} + a_{jki} - a_{kij} &= c_{ijk} \pm c_{ikj} + c_{jki} \pm c_{jik} - (c_{kij} \pm c_{kji}) \\ &= 2c_{ijk} - b_{ijk} - b_{kij} + b_{jki} \end{aligned}$$

where in the permutation of indices was used the remaining condition. □

Appendix B. An alternate variational problem for GR

The expression (6) for the variation of the connection taken as independent of the variation of the frame can be used to set an alternate variational principle for GR. In this case we can take as restriction EDS

$$\mathcal{I}'_{PG} := \langle \mathsf{tr}\,(\omega) \rangle_{\text{diff}}.$$

The annihilation of the variations (6) with respect to the connection yields to a set of Euler-Lagrange equations; namely, from

$$\left[\eta_{kp}\theta_{li} \wedge \left(\omega^{lp} + \omega^{pl}\right) + (-1)^{n+1} T^l \wedge \theta_{ikl}\right] \wedge \delta\omega^{ik} = 0$$

we obtain the equation of motion

$$\eta_{kp}\theta_{li} \wedge \left(\omega^{lp} + \omega^{pl}\right) + (-1)^{n+1} T^l \wedge \theta_{ikl} = 0. \tag{13}$$

From the skewsymmetry of θ_{ijk} it follows that

$$\eta_{kp}\theta_{li} \wedge \left(\omega^{lp} + \omega^{pl}\right) + \eta_{ip}\theta_{lk} \wedge \left(\omega^{lp} + \omega^{pl}\right) = 0.$$

The way to solve this system is very interesting, and uses Proposition 29.

Lemma 30. *Let* $x \mapsto m^{kl}$ *be a set of* 1*-forms on* M *such that* $\eta_{ij}m^{ij} = 0$. *If these forms solve the system*

$$\begin{cases} m^{pk} \wedge \left(\eta_{pq}\theta_{kl} \pm \eta_{pl}\theta_{kq}\right) = 0 \\ m^{pk} \mp m^{kp} = 0 \end{cases}$$

then

$$m^{ij} = 0.$$

Proof. Let us define the set of local $n-1$-forms

$$\theta_i := X_i \lrcorner \sigma_0,$$

where, as above, $\{X_1, \cdots, X_n\}$ is the frame dual to $\left\{\theta^1, \cdots, \theta^n\right\}$, and $\sigma_0 = \theta^1 \wedge \cdots \wedge \theta^n$. Then we have the identity

$$\theta^m \wedge \theta_{kl} = -\left(\delta^m_k \theta_l - \delta^m_l \theta_k\right).$$

Thus

$$\begin{aligned} m^{pk}\wedge\eta_{pq}\theta_{kl} &= \eta_{pq}m^{pk}_r\theta^r\wedge\theta_{kl} \\ &= -\eta_{pq}m^{pk}_r\left(\delta^r_k\theta_l-\delta^r_l\theta_k\right); \end{aligned}$$

therefore

$$\begin{aligned} 0 = m^{pk}\wedge\left(\eta_{pq}\theta_{kl}\pm\eta_{pl}\theta_{kq}\right) &= \\ = -m^{pk}_r\left[\eta_{pq}\left(\delta^r_k\theta_l-\delta^r_l\theta_k\right)\pm\eta_{pl}\left(\delta^r_k\theta_q-\delta^r_q\theta_k\right)\right] \\ = \eta_{pq}\left(-m^{pr}_r\theta_l+m^{pk}_l\theta_k\right)\pm\eta_{pl}\left(-m^{pr}_r\theta_q+m^{pk}_q\theta_k\right). \end{aligned}$$

By multiplying both sides of this identity by θ^l and adding up, we see that

$$\begin{aligned} 0 &= \theta^l\wedge\left[\eta_{pq}\left(-m^{pr}_r\theta_l+m^{pk}_l\theta_k\right)\pm\eta_{pl}\left(-m^{pr}_r\theta_q+m^{pk}_q\theta_k\right)\right] \\ &= \eta_{pq}\left(-nm^{pr}_r+m^{pr}_r\right)\sigma_0\pm\eta_{pq}\left(-m^{pr}_r\right)\sigma_0\pm\eta_{pk}m^{pk}_q\sigma_0 \end{aligned}$$

where it was used that

$$\theta^k\wedge\theta_l=\delta^k_l\sigma_0.$$

Then $\eta_{pq}\left(n-1\pm1\right)m^{pr}_r=\eta_{pk}m^{pk}_q=0$, and $m^{pr}_r=0$. Then we will have that

$$\left(\eta_{pq}m^{pk}_l\pm\eta_{pl}m^{pk}_q\right)\theta_k=0$$

and from here we can conclude that the system above can be written as

$$\begin{cases}\eta_{pq}m^{pk}_l\pm\eta_{pl}m^{pk}_q=0\\ m^{pk}_i\mp m^{kp}_i=0.\end{cases}$$

Therefore the set of unknowns $N_{ijk}:=\eta_{iq}\eta_{jp}m^{pq}_k$ solves the system

$$\begin{cases}N_{kql}\pm N_{klq}=0\\ N_{kpi}\mp N_{pki}=0.\end{cases}$$

Using proposition 29 we see that

$$N_{ijk}=0$$

is the unique solution for this system. □

By using the previous Lemma it follows that for a section to be an extremal section (under our choice $\mathrm{Tr}\,\omega=0$) it will be necessary that

$$\omega^{pl}+\omega^{lp}=0,$$

and, as a bonus, $T=0$. Thus the generators of the EDS $\mathcal{I}_{PG}$ are obtained as equations of motion.

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